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(Non)commutative Isotropization in Bianchi I with Barotropic Perfect Fluid and Λ Cosmological

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Abstract Using the WKB-like and Hamilton procedures classical solutions to commutative and noncommutative cosmology are found. This is carried out in the Bianchi type I cosmological model coupled to barotropic perfect fluid and cosmological constant. Noncommutativity is achieved by modifying the symplectic structure considering that minisuperspace variables do not commute, using a deformation between all the minisuperspace variables. It is shown that the anisotropic parameter $\beta_{\pm nc}$ for some value in the Λ cosmological term and noncommutative θ parameter, present a dynamical isotropization until a critical cosmic time t_c. After this time the effects of minisuperspace noncommutativity in the isotropization seem to disappear.

Keywords Noncommutative · Classical cosmology · Exact solutions

1 Introduction

Recently, a great interest has been generated in noncommutative spacetimes [1–4], mainly due to the fact that there are of strong motivations in the development of string and

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M-theories [5, 6]. A different approach to noncommutativity is through the introduction of noncommutative fields [7], that is, fields of their conjugate momenta are taken as noncommuting. There are several approaches in considering the notion of noncommutativity in cosmology, that could be the best alternative in the absence of a comprehensive and satisfactory theory from string theory. This analysis has been studied in many works [8, 9]. Here, taking coordinates as noncommuting, it has been shown that noncommutativity affects the spectrum of Cosmic Microwave Background. For example, in [8], noncommutative geometry suggest a non local inflaton field that changes the gaussianity and isotropy property of fluctuations. In cosmological systems, since the scale factors, matter fields and their conjugate momenta play the role of dynamical variables of the system, introduction of noncommutativity by adopting the noncommutativity between all fields, is particularly relevant. The simplest noncommutative classical and quantum cosmology of such models has been studied in different works [10–13].

On the other hand, there is a renewed interest on noncommutative theories to explain the appropriate modification of classical General Relativity, and hence of spacetime symmetries at short-distance scales, that implies modifications at large scales. General quantum mechanics arguments indicate that, it is not possible to measure a classical background spacetime at the Planck scale, due to the effects of gravitational backreaction [14]. It is therefore tempting to incorporate the dynamical features of spacetime at deeper kinematical level using the standard techniques of noncommutative classical field theory based in the so called Moyal product in which for all calculations purposes (differentiation, integration, etc.) the space time coordinates are treated as ordinary (commutative) variables and noncommutativity enters into play in the way in which fields are multiplied [15]. Using a modified symplectic structure on the space variables in the Hamilton approach, we assume that the minisuperspace variables do not commute, for this purpose we will modified the Poisson structure, this approach does not modify the Hamiltonian structure in the noncommutative fields. In the approach used, we choose that the momenta in both spaces are the same, $P_{q^\mu_{nc}} = P_{q^\mu}$, that is, they commute in both spaces.

Another way to extract useful dynamical information is through the WKB semiclassical approximation to the quantum Wheeler–DeWitt equation using the wave function $\Psi = e^{iS(q^\mu)}$. In this sense, we consider the usual approximation in the derivatives and the corresponding relation between the Einstein–Hamilton–Jacobi (EHJ) equation, it was possible to obtain classical solutions at the master equation found by this procedures. The classical field equations were checked for all solutions, using the REDUCE 3.8 algebraic packages.

The main idea in this paper is to find, the commutative $(\Omega, \beta_\pm)$ and noncommutative $(\Omega_{nc}, \beta_{\pm nc})$ classical solutions for the Bianchi Class A models, using two alternative approaches, known as WKB semiclassical approximation and Hamilton approach. From these solutions in the gauge $N = 1$ (the physical gauge), we can infer if the anysotropic parameters $\beta_{\pm nc}$ suffers changes toward isotropic ones (a constant curvature). This analysis is considered in particular with the Bianchi type I, coupled to barotropic perfect fluid and cosmological term.

The paper is then organized as follows. In Sect. 2, we obtain the WDW equation including the barotropic matter contribution, and the corresponding commutative classical solutions for the cosmological Bianchi type I, in the gauge $N = 1$, by the WKB semiclassical approximation and Hamilton procedure. Section 3 is devoted to the noncommutative classical solutions and the analysis of the isotropization is made too, in the physical gauge $N = 1$. Final remarks are presented in Sect. 4. For completeness, we can follow a similar prescription for the gauge $N = 24e^{3\Omega}$, where the corresponding solutions for both scenarios, the commutative, and noncommutative, are presented in Appendices 1 and 2, respectively.

Let us begin by recalling canonical formulation of the ADM formalism to the diagonal Bianchi Class A cosmological models. The metrics have the form

$$ds^2 = -(N^2 - N^j N_j)dt^2 + e^{2\Omega(t)} e^{2\beta_{ij}(t)}\, \omega^i \omega^j, \tag{1}$$

where N and N_i are the lapse and shift functions, respectively, $\Omega(t)$ is a scalar and $\beta_{ij}(t)$ a 3×3 diagonal matrix, $\beta_{ij} = \mathrm{diag}(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+)$, ω^i are one-forms that characterize each cosmological Bianchi type model, and that obey $d\omega^i = \frac{1}{2} C^i_{jk} \omega^j \wedge \omega^k$, C^i_{jk} the structure constants of the corresponding invariance group [16]. The metric for the Bianchi type I, takes the form

$$ds_I^2 = -N^2 dt^2 + e^{2\Omega} e^{2\beta_+ + 2\sqrt{3}\beta_-} dx^2 + e^{2\Omega} e^{2\beta_+ - 2\sqrt{3}\beta_-} dy^2 + e^{2\Omega} e^{-4\beta_+} dz^2. \tag{2}$$

The corresponding Lagrangian density is

$$L_{\mathrm{Total}} = \sqrt{-g}\,(R - 2\Lambda) + L_{\mathrm{matter}}, \tag{3}$$

and using (2), this has the following form

$$L = 6e^{3\Omega}\left[-\frac{\dot{\Omega}^2}{N} + \frac{\dot{\beta}_+^2}{N} + \frac{\dot{\beta}_-^2}{N} - \frac{\Lambda}{3}N + \frac{8}{3}\pi G N \rho\right], \tag{4}$$

where the overdot denotes time derivatives. The canonical momenta to coordinate fields are defined in the usual way

$$P_\Omega = \frac{\partial L}{\partial \dot{\Omega}} = -12 e^{3\Omega}\frac{\dot{\Omega}}{N}, \qquad P_+ = \frac{\partial L}{\partial \dot{\beta}_+} = 12 e^{3\Omega}\frac{\dot{\beta}_+}{N}, \qquad P_- = \frac{\partial L}{\partial \dot{\beta}_-} = 12 e^{3\Omega}\frac{\dot{\beta}_-}{N}, \tag{5}$$

and the correspondent Hamiltonian function is

$$H = \frac{N e^{-3\Omega}}{24}[-P_\Omega^2 + P_+^2 + P_-^2 - 48\Lambda e^{6\Omega} + 384\pi G M_\gamma e^{-3(\gamma-1)\Omega}] = 0, \tag{6}$$

together with barotropic state equation $p = \gamma\rho$, the Hamilton–Jacobi equation is obtained when we substitute $P_{q^\mu} \to \frac{dS_i}{dq^\mu}$ into (6). In what follows, we should consider the gauge $N = 1$.

2 Commutative Classical Solutions

2.1 Commutative Classical Solutions á la WKB

The quantum Wheeler–DeWitt (WDW) equation for these models is obtained by making the canonical quantization P_{q^μ} by $-i\partial_{q^\mu}$ in (6) with $q^\mu = (\Omega, \beta_+, \beta_-)$ is

$$\frac{e^{-3\Omega}}{24}\left[\frac{\partial^2}{\partial\Omega^2} - \frac{\partial^2}{\partial\beta_+^2} - \frac{\partial^2}{\partial\beta_-^2} - \lambda e^{6\Omega} + b_\gamma e^{-3(\gamma-1)\Omega}\right]\Psi = 0, \tag{7}$$

where $\lambda = 48\Lambda$, $b_\gamma = 384\pi G M_\gamma$. We now proceed to apply the WKB semiclassical approximation using the ansatz

$$\Psi(\Omega, \beta_\pm) = e^{i[S_1(\Omega) + S_2(\beta_+) + S_3(\beta_-)]}, \tag{8}$$

into (7), and without any loss of generality, one can consider the condition $\frac{d^2S_i}{dq_i^2}$ be small i.e.,

$$\left(\frac{dS_1}{d\Omega}\right)^2 \gg \frac{d^2S_1}{d\Omega^2}, \qquad \left(\frac{dS_2}{d\beta_+^2}\right)^2 \gg \frac{d^2S_2}{d\beta_+^2}, \qquad \left(\frac{dS_2}{d\beta_-^2}\right)^2 \gg \frac{d^2S_2}{d\beta_-^2}, \tag{9}$$

to get the classical Einstein–Hamilton–Jacobi equation

$$-\left(\frac{dS_1}{d\Omega}\right)^2 + \left(\frac{dS_2}{d\beta_+}\right)^2 + \left(\frac{dS_3}{d\beta_-}\right)^2 - \lambda e^{6\Omega} + be^{-3(\gamma-1)\Omega} = 0, \tag{10}$$

which can be separated in a set of differential equations

$$-\left(\frac{dS_1}{d\Omega}\right)^2 + a_1^2 - \lambda e^{6\Omega} + be^{-3(\gamma-1)\Omega} = 0, \tag{11}$$

$$\left(\frac{dS_2}{d\beta_+}\right)^2 = n_1^2, \tag{12}$$

$$\left(\frac{dS_3}{d\beta_-}\right)^2 = p_1^2, \tag{13}$$

where a_1^2, n_1^2 and p_1^2 are the separation constants and their relations is $a_1^2 = n_1^2 + p_1^2$. Therefore using the relations between (5, 11–13) we have the following equations of motion

$$\pm\sqrt{a_1^2 - \lambda e^{6\Omega} + b_\gamma e^{-3(\gamma-1)\Omega}} \equiv -12e^{3\Omega}\frac{\dot{\Omega}}{N}, \tag{14}$$

$$\pm n_1 \equiv 12e^{3\Omega}\frac{\dot{\beta}_+}{N}, \tag{15}$$

$$\pm p_1 \equiv 12e^{3\Omega}\frac{\dot{\beta}_-}{N}. \tag{16}$$

The main master equation to solved in the gauge $N = 1$, is

$$\frac{dt}{12} = \frac{d\Omega}{\sqrt{a_1^2 e^{-6\Omega} + b_\gamma e^{-3(\gamma+1)\Omega} - \lambda}}, \tag{17}$$

the other two equations (15) and (16) are trivially integrable. For particular stadium of the universe evolution, given by the γ parameter, we present these classical solutions in Table 1.

2.2 Classical Solutions via Hamiltonian Formalism

In order to find the commutative equation of motion, we use the classical phase space variables $(\Omega, \beta_\pm)$, where the Poisson algebra for these minisuperspace variables are

$$\{\Omega, \beta_\pm\} = \{\beta_+, \beta_-\} = \{P_\Omega, P_\pm\} = \{P_+, P_-\} = 0, \qquad \{q^\mu, P_{q^\mu}\} = 1, \tag{18}$$

and recalling the Hamiltonian equation (6), we obtain the classical solutions with the following procedure.

Table 1 Classical solutions for $\gamma = -1, 1, 0$, and constraints q, a_1 and b_0

Case	Commutative solutions	
$\gamma = -1,\ \Lambda \neq 0,\ \rho_{-1} = M_{-1}$	$\Omega = \frac{1}{3}\mathrm{Ln}[\frac{e^{2qt}-4a_1^2}{16qe^{qt}}]$, $\beta_+ = \pm\frac{2}{3}\frac{n_1}{a_1}\mathrm{arctanh}[\frac{e^{qt}}{2a_1}]$, $\beta_- = \pm\frac{2}{3}\frac{p_1}{a_1}\mathrm{arctanh}[\frac{e^{qt}}{2a_1}]$	$q^2 = 24\pi G M_{-1} - 3\Lambda$, $a_1^2 = n_1^2 + p_1^2$,
$\gamma = 1,\ \Lambda < 0,\ \rho_1 = M_1 e^{-6\Omega}$	$\Omega = \frac{1}{3}\mathrm{Ln}[\frac{e^{2qt}-4a_1^2}{16qe^{qt}}]$, $\beta_+ = \pm\frac{2}{3}\frac{n_1}{a_1}\mathrm{arctanh}[\frac{e^{qt}}{2a_1}]$, $\beta_- = \pm\frac{2}{3}\frac{p_1}{a_1}\mathrm{arctanh}[\frac{e^{qt}}{2a_1}]$	$q = \sqrt{3\lvert\Lambda\rvert}$, $a_1^2 = n_1^2 + p_1^2 + 384\pi G M_1$,
$\gamma = 1,\ \Lambda = 0,\ \rho_1 = M_1 e^{-6\Omega}$	$\Omega = \frac{1}{3}\mathrm{Ln}[\frac{a_1}{4}t]$, $\beta_+ = \pm\mathrm{Ln}[t^{-\frac{n_1}{3a_1}}]$, $\beta_- = \pm\mathrm{Ln}[t^{-\frac{p_1}{3a_1}}]$	$a_1^2 = n_1^2 + p_1^2 + 384\pi G M_1$,
$\gamma = 0,\ \Lambda = 0,\ \rho_0 = M_0 e^{-3\Omega}$	$\Omega = \frac{1}{3}\mathrm{Ln}[\frac{b_0 t^2}{64} + \frac{a_1 t}{4}]$, $\beta_+ = \pm\frac{n_1}{3a_1}\mathrm{Ln}[\frac{16a_1+b_0 t}{t}]$, $\beta_- = \pm\frac{p_1}{3a_1}\mathrm{Ln}[\frac{16a_1+b_0 t}{t}]$	$b_0 = 384\pi G M_0$, $a_1^2 = n_1^2 + p_1^2$,

The classical equations of motion for the phase variables Ω, $\beta_\pm$, $P_\pm$, and P_Ω are

$$\dot{\Omega} = \{\Omega, H\} = -\frac{1}{12}e^{-3\Omega}P_\Omega, \tag{19}$$

$$\dot{\beta_-} = \{\beta_-, H\} = \frac{1}{12}e^{-3\Omega}P_-, \tag{20}$$

$$\dot{\beta_+} = \{\beta_+, H\} = \frac{1}{12}e^{-3\Omega}P_+, \tag{21}$$

$$\dot{P}_\Omega = \{P_\Omega, H\} = \frac{1}{8}e^{-3\Omega}[-P_\Omega^2 + P_-^2 + P_+^2 + \lambda e^{6\Omega} + \gamma b_\gamma e^{-3(\gamma-1)\Omega}], \tag{22}$$

$$\dot{P}_- = \{P_-, H\} = 0 \quad \to \quad P_- = \pm p_1 = \text{const.}, \tag{23}$$

$$\dot{P}_+ = \{P_+, H\} = 0 \quad \to \quad P_+ = \pm n_1 = \text{const.} \tag{24}$$

Introducing (6) into (22), we have

$$8e^{-3\Omega}\dot{P}_\Omega = 2\lambda + (\gamma - 1)b_\gamma e^{-3(\gamma+1)\Omega}, \tag{25}$$

which can be integrated to obtain the relation for P_Ω

$$P_\Omega = \pm\sqrt{a_1^2 - \lambda e^{6\Omega} + b_\gamma e^{-3(\gamma-1)\Omega}}, \tag{26}$$

where $a_1^2 = n_1^2 + p_1^2$.

The set of equations (19–21) is equivalent to the set of equations (14–16), equations used to obtain the classical solutions.

Just to remark, the solutions obtained with the Hamiltonian formalism and the WKB-like procedure are equivalent to solving GR field equations.

3 Noncommutative Solutions

Let us begin introducing the noncommutative deformation of the minisuperspace [10] in the WDW equation, this time, between all the variables of the minisuperspace, assuming that Ω_{nc} and $\beta_{\pm nc}$ obey the commutation relation

$$[\Omega_{nc}, \beta_{-nc}] = i\theta_1, \qquad [\Omega_{nc}, \beta_{+nc}] = i\theta_2, \qquad [\beta_{-nc}, \beta_{+nc}] = i\theta_3. \tag{27}$$

Instead of working directly with the physical variables Ω and $\beta_\pm$ we may achieve all the above solutions by making use of the auxiliary canonical variables Ω_{nc} and $\beta_{\pm nc}$ defined as

$$\Omega_{nc} \equiv \Omega - \frac{\theta_1}{2} P_- - \frac{\theta_2}{2} P_+, \tag{28}$$

$$\beta_{-nc} \equiv \beta_- + \frac{\theta_1}{2} P_\Omega - \frac{\theta_3}{2} P_+, \tag{29}$$

$$\beta_{+nc} \equiv \beta_+ + \frac{\theta_2}{2} P_\Omega + \frac{\theta_3}{2} P_-. \tag{30}$$

Maintaining the usual commutation relations between the fields, i.e., $[q^\mu, q^\nu] = 0$. A shift generalization for the commutative symplectic structure can be made it through the change

$$q^\mu \equiv q^\mu_{nc} + \frac{1}{2}\theta^{\mu\nu} P_\nu, \tag{31}$$

where $\theta^{\mu\nu}$ is an antisymmetric matrix, and the identifications $P_\Omega = P_{\Omega nc}$ and $P_\pm = P_{\pm nc}$. With this shift and the usual canonical quantization $P_{q^\mu} \to -i\partial_{q^\mu}$, we arrive to the noncommutative WDW equation

$$\left[\frac{\partial^2}{\partial\Omega^2_{nc}} - \frac{\partial^2}{\partial\beta^2_{+nc}} - \frac{\partial^2}{\partial\beta^2_{-nc}} - \lambda e^{6\Omega_{nc}} + b_\gamma e^{-3(\gamma-1)\Omega_{nc}}\right]\Psi(\Omega, \beta_\pm) = 0, \tag{32}$$

where $\lambda = 48\Lambda$, $b_\gamma = 384\pi G M_\gamma$. At this point we have a noncommutative WDW equation and noncommutative Hamiltonian. In what follows, we shall consider a wave function and apply the WKB procedure to obtain classical solutions.

3.1 Noncommutative Classical Solutions á la WKB

In order to find noncommutative classical solutions through the WKB approximation, we use the fact that $e^{i\theta\frac{\partial}{\partial x}} e^{\eta x} \equiv e^{i\eta\theta} e^{\eta x}$, and the ansatz for the wavefunction $\Psi(\Omega_{nc}, \beta_{\pm nc}) = e^{i[S_1(\Omega_{nc}) \pm n_1\beta_{+nc} \pm p_1\beta_{-nc}]}$, where we use explicitly $S_2(\beta_{+nc}) = \pm n_1\beta_{+nc}$ and $S_3(\beta_{-nc}) = \pm p_1\beta_{-nc}$ to get the classical noncommutative Einstein–Hamilton–Jacobi (EHJ) equation

$$-\left(\frac{dS_1}{d\Omega_{nc}}\right)^2 + \left(\frac{dS_2}{d\beta_{+nc}}\right)^2 + \left(\frac{dS_3}{d\beta_{-nc}}\right)^2 - \lambda e^{6\Omega_{nc}} + b e^{-3(\gamma-1)\Omega_{nc}} = 0, \tag{33}$$

which can be separated in a set of differential equations with $m_1^2 = n_1^2 + p_1^2$. We have the following noncommutative equations of motion

Table 2 Noncommutative solutions for, $\gamma=-1,1,0$, and constraints q, a_1 and b_0

Case	Noncommutative solutions
$\gamma=-1,\ \Lambda\neq 0,\ \rho_{-1}=M_{-1}$, $a_1^2=n_1^2+p_1^2$, $q^2=24\pi GM_{-1}-3\Lambda$	$\Omega_{nc}=\frac{1}{3}\mathrm{Ln}[\frac{e^{2qt}-4a_1^2}{16qe^{qt}}]-\frac{\theta_1}{2}p_1-\frac{\theta_2}{2}n_1$, $\beta_{+nc}=\pm\frac{2}{3}\frac{n_1}{a_1}\mathrm{arctanh}[\frac{e^{qt}}{2a_1}]+\frac{\theta_2}{8}(\frac{e^{qt}}{4}+a_1^2e^{-qt})-\frac{\theta_3}{2}p_1$, $\beta_{-nc}=\pm\frac{2}{3}\frac{p_1}{a_1}\mathrm{arctanh}[\frac{e^{qt}}{2a_1}]+\frac{\theta_1}{8}(\frac{e^{qt}}{4}+a_1^2e^{-qt})+\frac{\theta_3}{2}n_1$
$\gamma=1,\ \Lambda<0,\ \rho_1=M_1e^{-6\Omega}$, $a_1^2=n_1^2+p_1^2+384\pi GM_1$	$\Omega_{nc}=\frac{1}{3}\mathrm{Ln}[\frac{e^{2qt}-4a_1^2}{16qe^{qt}}]-\frac{\theta_1}{2}p_1-\frac{\theta_2}{2}n_1,\quad q=\sqrt{3\vert\Lambda\vert}$, $\beta_{+nc}=\pm\frac{2}{3}\frac{n_1}{a_1}\mathrm{arctanh}[\frac{e^{qt}}{2a_1}]+\frac{\theta_2}{8}(\frac{e^{qt}}{4}+a_1^2e^{-qt})-\frac{\theta_3}{2}p_1$, $\beta_{-nc}=\pm\frac{2}{3}\frac{p_1}{a_1}\mathrm{arctanh}[\frac{e^{qt}}{2a_1}]+\frac{\theta_1}{8}(\frac{e^{qt}}{4}+a_1^2e^{-qt})+\frac{\theta_3}{2}n_1$
$\gamma=1,\ \Lambda=0,\ \rho_1=M_1e^{-6\Omega}$, $a_1^2=n_1^2+p_1^2+384\pi GM_1$	$\Omega_{nc}=\frac{1}{3}\mathrm{Ln}[\frac{a_1}{4}t]-\frac{\theta_1}{2}p_1-\frac{\theta_2}{2}n_1$, $\beta_{+nc}=\pm\mathrm{Ln}[t^{-\frac{n_1}{3a_1}}]+\frac{\theta_2}{2}a_1-\frac{\theta_3}{2}p_1$, $\beta_{-nc}=\pm\mathrm{Ln}[t^{-\frac{p_1}{3a_1}}]+\frac{\theta_1}{2}a_1+\frac{\theta_3}{2}n_1$
$\gamma=0,\ \Lambda=0,\ \rho_0=M_0e^{-3\Omega}$, $b_0=384\pi GM_0$, $a_1^2=n_1^2+p_1^2$	$\Omega_{nc}=\frac{1}{3}\mathrm{Ln}[\frac{b_0t^2}{64}+\frac{a_1t}{4}]-\frac{\theta_1}{2}p_1-\frac{\theta_2}{2}n_1$, $\beta_{+nc}=\pm\frac{n_1}{3a_1}\mathrm{Ln}[\frac{16a_1+b_0t}{t}]+\frac{\theta_2}{2}\sqrt{a_1^2+\frac{b_0t^2}{64}+\frac{a_1t}{4}}-\frac{\theta_3}{2}p_1$, $\beta_{-nc}=\pm\frac{p_1}{3a_1}\mathrm{Ln}[\frac{16a_1+b_0t}{t}]+\frac{\theta_1}{2}\sqrt{a_1^2+\frac{b_0t^2}{64}+\frac{a_1t}{4}}+\frac{\theta_3}{2}n_1$

$$\pm\sqrt{a_1^2-\lambda e^{6\Omega_{nc}}+b_\gamma e^{-3(\gamma-1)\Omega_{nc}}}\equiv -12e^{3\Omega_{nc}}\frac{\dot{\Omega}_{nc}}{N}, \tag{34}$$

$$\pm n_1\equiv 12e^{3\Omega_{nc}}\frac{\dot{\beta}_{+nc}}{N}, \tag{35}$$

$$\pm p_1\equiv 12e^{3\Omega_{nc}}\frac{\dot{\beta}_{-nc}}{N}. \tag{36}$$

One just need to be careful in (34–36), and apply the chain rule to the variables (28–30), in order to get the right solution, $\dot{\beta}_{-nc}=\frac{\partial\beta_-}{\partial t}+\frac{\partial\beta_{-nc}}{\partial P_\Omega}\frac{\partial p_\Omega}{\partial t}+\frac{\partial\beta_{-nc}}{\partial P_+}\frac{\partial p_+}{\partial t}+\frac{\partial\beta_{-nc}}{\partial P_-}\frac{\partial p_-}{\partial t}=\dot{\beta}_-+\frac{\theta_1}{2}\dot{P}_\Omega$. In this sense, all solutions to find in the commutative case, remain for the noncommutative case with the corresponding shift, as we show in Table 2.

As an example we present in Fig. 1, the behavior of $\beta_{\pm nc}$ in the case $\gamma=-1$, and $\Lambda<0$ (the sign yet was taken in account in the corresponding equation) shown in Table 2. We can see that the anisotropic parameters $\beta_{\pm nc}$ for some value in the λ cosmological term and noncommutative θ parameter, present a dynamical isotropization until a critical cosmic time t_c. This defines a type of noncommutative cosmic scale, for which effects of noncommutativity in the anisotropies can be relevant. After this time the effects of minisuperspace noncommutativity in the isotropization seem to disappear.

3.2 Noncommutative Classical Solutions á la Hamilton

In the commutative model we know that the solutions to Hamilton's equations are the same as in General Relativity. Now the natural extension is to consider the noncommutative version of our model, with the idea of noncommutativity between the three variables $(\Omega_{nc},\beta_{\pm nc})$, so we apply a deformation of the Poisson algebra. For this we start with the usual Hamiltonian (6), but the symplectic structure is modified as follows

$$\{P_\Omega,P_\pm\}_\star=\{P_+,P_-\}_\star=0,\qquad \{q^\mu,P_{q^\mu}\}_\star=1, \tag{37}$$

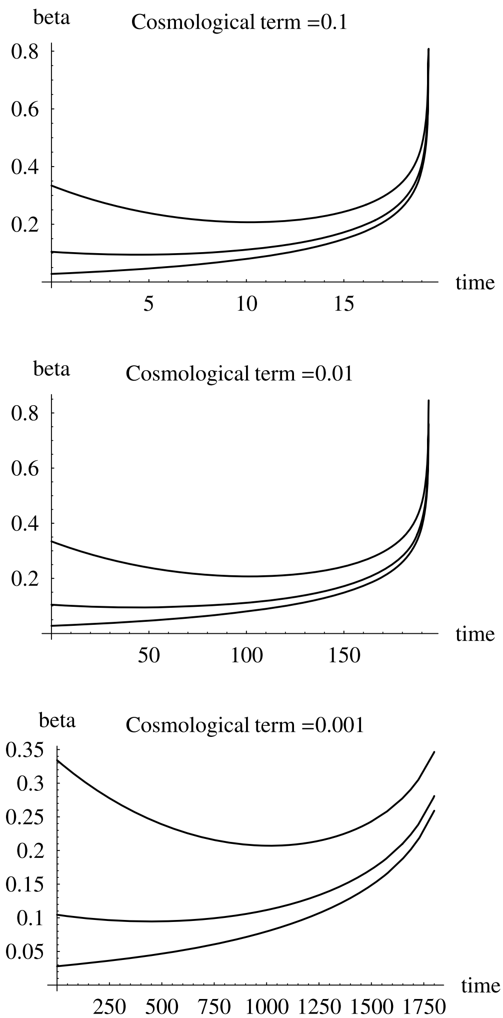

Fig. 1 Plots of $\beta_{\pm nc}$ that appear in the second line in the Table 2, using the values in the parameters $n_1 = 1$, $p_1 = 1$, $b_0 = 10$ and $\theta = 0, 0.05, 0.2$, from *bottom* to *top* in the figure. The possible isotropization is seen in function of the curvature, but it appears again in this fields after critical cosmic time t_c

$$\{\Omega, \beta_-\}_\star = \theta_1, \qquad \{\Omega, \beta_+\}_\star = \theta_2, \qquad \{\beta_-, \beta_+\}_\star = \theta_3, \tag{38}$$

where the $\star$ is the Moyal product [15]. In the second case, the Hamiltonian is modified by the shift (28–30) resulting

$$H_{nc} = \frac{Ne^{-3\Omega_{nc}}}{24}[-P_\Omega^2 + P_+^2 + P_-^2 - \lambda e^{6\Omega_{nc}} + b_\gamma e^{-3(\gamma-1)\Omega_{nc}}] = 0, \tag{39}$$

but the symplectic structure is the one that we know, the commutative one (18).

The noncommutative equations of motion, for the first formalism that we exposed have the original variables, but with the modified symplectic structure,

$$\begin{aligned} \dot{q}^\mu_{nc} &= \{q^\mu, H\}_\star, \\ \dot{P}^\mu_{nc} &= \{P^\mu, H\}_\star, \end{aligned} \tag{40}$$

and for the second formalism we use the shifted variables but with the original (commutative) symplectic structure

$$
\begin{aligned}
\dot{q}^{\mu}_{nc} &= \{q^{\mu}_{nc}, H_{nc}\}, \\
\dot{P}^{\mu}_{nc} &= \{P^{\mu}_{nc}, H_{nc}\},
\end{aligned}
\tag{41}
$$

in both approaches we have the same result. Therefore the equations of motion take the form

$$\dot{\Omega}_{nc} = \{\Omega, H\}_\star = \{\Omega_{nc}, H_{nc}\} = -\frac{e^{-3\Omega_{nc}}}{12} P_\Omega, \tag{42}$$

$$\dot{\beta}_{-nc} = \{\beta_-, H\}_\star = \{\beta_{-nc}, H_{nc}\} = \frac{e^{-3\Omega_{nc}}}{12} P_- + \frac{\theta_1}{2} \dot{P}_\Omega, \tag{43}$$

$$\dot{\beta}_{+nc} = \{\beta_+, H\}_\star = \{\beta_{+nc}, H_{nc}\} = \frac{e^{-3\Omega_{nc}}}{12} P_+ + \frac{\theta_2}{2} \dot{P}_\Omega, \tag{44}$$

$$\dot{P}_\Omega = \{P_\Omega, H\}_\star = \{P_\Omega, H_{nc}\} = \frac{e^{-3\Omega_{nc}}}{8} [6\lambda e^{6\Omega_{nc}} + 3(\gamma - 1) b_\gamma e^{-3(\gamma-1)\Omega_{nc}}], \tag{45}$$

$$\dot{P}_- = \{P_-, H\}_\star = \{P_-, H_{nc}\} = 0 \quad \rightarrow \quad P_- = p_1, \tag{46}$$

$$\dot{P}_+ = \{P_+, H\}_\star = \{P_+, H_{nc}\} = 0 \quad \rightarrow \quad P_+ = n_1, \tag{47}$$

if we proceed as in the commutative case we get the solutions showed in Table 2.

4 Conclusions

In this work by using the equivalence between General Relativity, the WKB-approximation and Hamiltonian formalism, noncommutative scenarios are constructed. This was achieved by deforming the minisuperspace for the Bianchi type I cosmological model coupled to barotropic perfect fluid and cosmological term. in the gauge $N = 1$, as we can see the solution Ω_{nc} is the commutative solution plus a function on θ_i, independent of time. We have also analyzed the $\beta_{\pm nc}$ noncommutative solutions, in some ranges on the parameter θ_i and cosmological constant, occurs a dynamical isotropization, i.e., $\beta_{nc} \rightarrow$ a constant curvature, noting that different evolution with respect to the commutative $\beta_\pm$, but after a critical time t_c the evolution is the same in both scenarios. For completeness the solutions in the gauge $N = 24e^{3\Omega}$ (see Appendices 1 and 2) are presented, one of the advantages of this gauge is that the solutions are very simple, this is something to take in to account when we introduce a more complex form of matter, where in the gauge $N(t) = 1$ analytical solutions can not be found. This approach can be used for other Bianchi cosmological model, which will be reported elsewhere.

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Appendix 1 Commutative Classical Solutions in the Gauge $N = 24e^{3\Omega}$

In this appendix we present the classical solutions in the gauge $N = 24e^{3\Omega}$; the equations are much simpler to solve in this gauge.

Table 3 Classical solutions for $\gamma = -1, \frac{1}{3}, 1, 0$, and constraints a_1, b_0 and b_1

Case	Commutative solutions	
$\gamma = -1,\ \Lambda \neq 0,\ \rho_{-1} = M_{-1}$	$\Omega = \frac{1}{6}\mathrm{Ln}[-\frac{a_1^2}{384\pi G M_{-1} - 48\Lambda}\mathrm{Sech}^2(6a_1 t)]$, $\beta_+ = \pm 2n_1 t$, $\beta_- = \pm 2p_1 t$	$a_1^2 = n_1^2 + p_1^2$,
$\gamma = 1,\ \Lambda \neq 0,\ \rho_1 = M_1 e^{-6\Omega}$	$\Omega = \frac{1}{6}\mathrm{Ln}[\frac{a_1^2}{48\Lambda}\mathrm{Sech}^2(6a_1 t)$, $\beta_+ = \pm 2n_1 t$, $\beta_- = \pm 2p_1 t$	$a_1^2 = n_1^2 + p_1^2 + 384\pi G M_1$,
$\gamma = 1,\ \Lambda = 0,\ \rho_1 = M_1 e^{-6\Omega}$	$\Omega = 2\sqrt{a_1^2 + b_1}\,t$, $\beta_+ = -2n_1 t$, $\beta_- = -2p_1 t$	$a_1^2 = n_1^2 + p_1^2 + 384\pi G M_1$,
$\gamma = 0,\ \Lambda = 0,\ \rho_0 = M_0 e^{-3\Omega}$	$\Omega = \frac{1}{3}\mathrm{Ln}[-\frac{a_1^2}{b_0}\mathrm{Sech}^2(3a_1 t)]$, $\beta_+ = \pm 2n_1 t$, $\beta_- = \pm 2p_1 t$	$b_0 = 384\pi G M_0$, $a_1^2 = n_1^2 + p_1^2$,
$\gamma = \frac{1}{3},\ \Lambda = 0,\ \rho_0 = M_{\frac{1}{3}} e^{-4\Omega}$	$\Omega = \frac{1}{2}\mathrm{Ln}[-\frac{a_1^2}{b_{\frac{1}{3}}}\mathrm{Sech}^2(2a_1 t)]$, $\beta_+ = \pm 2n_1 t$, $\beta_- = \pm 2p_1 t$	$a_1^2 = n_1^2 + p_1^2$, $b_{\frac{1}{3}} = 384\pi G M_{\frac{1}{3}}$,

1.1 Commutative Classical Solutions á la WKB

The master equation becomes

$$2dt = \frac{d\Omega}{\sqrt{a_1^2 - \lambda e^{6\Omega} + b_\gamma e^{-3(\gamma-1)\Omega}}}, \tag{48}$$

and the other two equations are immediately integrable. For particular cases of γ parameter, we present the classical solutions in Table 3

1.2 Classical Solutions via Hamiltonian formalism

With the gauge fixed to $N = 24e^{3\Omega}$ we can see that the Hamiltonian takes the form

$$H = -P_\Omega^2 + P_+^2 + P_-^2 - \lambda e^{6\Omega} + b_\gamma e^{-3(\gamma-1)\Omega} = 0. \tag{49}$$

The Poisson brackets structure yields equations of motion

$$\dot{\Omega} = \{\Omega, H\} = -2P_\Omega, \tag{50}$$

$$\dot{\beta_-} = \{\beta_-, H\} = 2P_- \quad \rightarrow \quad \beta_- = \pm 2p_1 t, \tag{51}$$

$$\dot{\beta_+} = \{\beta_+, H\} = 2P_+ \quad \rightarrow \quad \beta_+ = \pm 2n_1 t, \tag{52}$$

$$\dot{P}_\Omega = \{P_\Omega, H\} = [+6\lambda e^{6\Omega} + 3(\gamma - 1)b_\gamma e^{-3(\gamma-1)\Omega}], \tag{53}$$

$$\dot{P}_- = \{P_-, H\} = 0 \quad \to \quad P_- = \pm p_1 = \text{const.}, \tag{54}$$

$$\dot{P}_+ = \{P_+, H\} = 0 \quad \to \quad P_+ = \pm n_1 = \text{const.} \tag{55}$$

Using (49), introducing (54) and (55), we obtain the expression for P_Ω:

$$P_\Omega = \sqrt{m_1^2 - \lambda e^{6\Omega} + b_\gamma e^{-3(\gamma-1)\Omega}}, \tag{56}$$

being self-consistent with (53), where $a_1^2 = n_1^2 + p_1^2$. Introducing this equation into (50) we get the master equation found to solve the Einstein field equation in this gauge, where the classical solutions are presented in Table 3.

Appendix 2 Noncommutative Classical Solutions

Table 4 Noncommutative solutions for $\gamma = -1, \frac{1}{3}, 1, 0$, and constraints a_1, b_0 and b_1

Case	Noncommutative solutions
$\gamma = -1,\ \Lambda \neq 0,\ \rho_{-1} = M_{-1}$, $a_1^2 = n_1^2 + p_1^2$	$\Omega_{nc} = \frac{1}{6}\text{Ln}[-\frac{a_1^2}{384\pi G M_{-1} - 48\Lambda}\text{Sech}^2(6a_1 t)] - \frac{\theta_1}{2}p_1 - \frac{\theta_2}{2}n_1$, $\beta_{+nc} = \pm 2n_1 t + \frac{\theta_2 a_1}{2}\tanh(6a_1 t) - \frac{\theta_3}{2}p_1$, $\beta_{-nc} = \pm 2p_1 t + \frac{\theta_1 a_1}{2}\tanh(6a_1 t) + \frac{\theta_3}{2}n_1$
$\gamma = 1,\ \Lambda \neq 0,\ \rho_1 = M_1 e^{-6\Omega}$, $a_1^2 = n_1^2 + p_1^2 + 384\pi G M_1$	$\Omega_{nc} = \frac{1}{6}\text{Ln}[\frac{a_1^2}{48\Lambda}\text{Sech}^2(6a_1 t)] - \frac{\theta_1}{2}p_1 - \frac{\theta_2}{2}n_1$, $\beta_{+nc} = \pm 2n_1 t + \frac{\theta_2 a_1}{2}\tanh(6a_1 t) - \frac{\theta_3}{2}p_1$, $\beta_{-nc} = \pm 2p_1 t + \frac{\theta_1 a_1}{2}\tanh(6a_1 t) + \frac{\theta_3}{2}n_1$
$\gamma = 1,\ \Lambda = 0,\ \rho_1 = M_1 e^{-6\Omega}$, $a_1^2 = n_1^2 + p_1^2 + 384\pi G M_1$	$\Omega_{nc} = 2a_1 t - \frac{\theta_1}{2}p_1 - \frac{\theta_2}{2}n_1$, $\beta_{+nc} = -2n_1 t + \frac{\theta_2}{2}a_1 - \frac{\theta_3}{2}p_1$, $\beta_{-nc} = -2p_1 t + \frac{\theta_1}{2}a_1 + \frac{\theta_3}{2}n_1$
$\gamma = 0,\ \Lambda = 0,\ \rho_0 = M_0 e^{-3\Omega}$, $b_0 = 384\pi G M_0$, $a_1^2 = n_1^2 + p_1^2$	$\Omega_{nc} = \frac{1}{3}\text{Ln}[-\frac{a_1^2}{b_0}\text{Sech}^2(3a_1 t)] - \frac{\theta_1}{2}p_1 - \frac{\theta_2}{2}n_1$, $\beta_{+nc} = \pm 2n_1 t + \frac{\theta_2 a_1}{2}\tanh(3a_1 t) - \frac{\theta_3}{2}p_1$, $\beta_{-nc} - = \pm 2p_1 t + \frac{\theta_1 a_1}{2}\tanh(3a_1 t) + \frac{\theta_3}{2}n_1$
$\gamma = \frac{1}{3},\ \Lambda = 0,\ \rho_0 = M_{\frac{1}{3}} e^{-4\Omega}$, $a_1^2 = n_1^2 + p_1^2$	$\Omega_{nc} = \frac{1}{2}\text{Ln}[-\frac{a_1^2}{b_{\frac{1}{3}}}\text{Sech}^2(2a_1 t)] - \frac{\theta_1}{2}p_1 - \frac{\theta_2}{2}n_1$, $\beta_{+nc} = \pm 2n_1 t + \frac{\theta_2 a_1}{2}\tanh(2a_1 t) - \frac{\theta_3}{2}p_1$, $\beta_{-nc} = \pm 2p_1 t + \frac{\theta_1 a_1}{2}\tanh(2a_1 t) + \frac{\theta_3}{2}n_1$

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